The stability problem and special solutions for the 5-components Maxwell-Bloch equations

Petre Birtea, Ioan Caşu*

Departamentul de Matematică, Universitatea de Vest din Timişoara

Bd. V. Pârvan, Nr. 4, 300223 Timişoara, România

E-mail: birtea@math.uvt.ro; casu@math.uvt.ro

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Abstract

For the 5-components Maxwell-Bloch system the stability problem for the isolated equilibria is completely solved. Using the geometry of the symplectic leaves, a detailed construction of the homoclinic orbits is given. Studying the problem of invariant sets for the system, we discover a rich family of periodic solutions in explicit form.

1 Introduction

After averaging and neglecting non-resonant terms, the unperturbed Maxwell-Bloch dynamics in the rotating wave approximation (RWA) is given by

$$\begin{cases} \dot{X} = Y \\ \dot{Y} = XZ \\ \dot{Z} = -\frac{1}{2}(XY^* + X^*Y), \end{cases}$$

where X, Y are complex scalar functions, that are denoting the self-consistent electric field and respectively the polarizability of the laser-matter, Z is a real scalar function, which denotes the difference of its occupation numbers. The superscript * stands for the complex conjugate. For more details about the history and physical interpretations of this system see [9], [10], [11].

Writing $X = x_1 + \imath x_2, Y = y_1 + \imath y_2$ and $Z = z$ the above system transforms into the 5-components Maxwell-Bloch system

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = x_1 z \\ \dot{x}_2 = y_2 \\ \dot{y}_2 = x_2 z \\ \dot{z} = -(x_1 y_1 + x_2 y_2). \end{cases} \tag{1.1}$$

The Maxwell-Bloch system in the form (1.1) has the advantage of a rich underlying geometrical structure that can be used in the study of its dynamical behavior.

The system (1.1) admits a Hamilton-Poisson formulation, where the Poisson tensor is given by

$$J(x_1, y_1, x_2, y_2, z) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & x_2 \\ 0 & -x_1 & 0 & -x_2 & 0 \end{bmatrix}$$

*Tel/fax: +40740928382/+40256592316; E-mail: casu@math.uvt.ro (Corresponding author)

and the Hamiltonian function is given by

$$H(x_1, y_1, x_2, y_2, z) = \frac{1}{2}(y_1^2 + y_2^2 + z^2).$$

The system has two additional constants of motion, namely the Casimir of the Poisson structure J, given by

$$C(x_1, y_1, x_2, y_2, z) = \frac{1}{2}(x_1^2 + x_2^2) + z$$

and a constant of motion derived from a bi-Hamiltonian structure of the system (1.1) (see [10]) given by

$$I(x_1, y_1, x_2, y_2, z) = x_2 y_1 - x_1 y_2.$$

A commuting property of the constants of motion H and I holds, i.e. $\{H, I\} = 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket associated to the Poisson tensor J.

2 Stability of equilibria

By a direct computation we obtain three families of equilibria for the system (1.1):

$$\mathcal{E}_1 = \{(0,0,0,0,M) | \ M \in \mathbb{R}^*\}; \quad \mathcal{E}_2 = \{(M,0,N,0,0) | \ M, N \in \mathbb{R}, M^2 + N^2 \neq 0\}; \quad \mathcal{E}_3 = \{(0,0,0,0,0)\}.$$

It is a well known fact that the dynamics of a Hamilton-Poisson system is foliated by the symplectic leaves associated to the Poisson structure. In our case the regular symplectic leaves are given by the connected components corresponding to pre-images of regular values of the Casimir function C. We denote by $\mathcal{O}_c = C^{-1}(c), c \in \mathbb{R}$ the regular symplectic leaves of the Poisson structure J.

The restriction of the dynamics (1.1) to a regular leaf $\mathcal{O}_c$ becomes a completely integrable Hamiltonian system

$$(\mathcal{O}_c, \omega_{\mathcal{O}_c}, H|_{\mathcal{O}_c}), \tag{2.1}$$

where the second commuting constant of motion is $I|_{\mathcal{O}_c}$. We will study the stability problem of equilibria on regular leaves $\mathcal{O}_c$ analogously to the approach used in [2].

The equilibria of the Hamiltonian system (2.1) can be divided in two types:

$$\mathcal{K}_0 := \{(x_1, y_1, x_2, y_2, z) \in \mathcal{O}_c \mid \mathbf{d}\left(H|_{\mathcal{O}_c}\right)(x_1, y_1, x_2, y_2, z) = 0, \ \mathbf{d}\left(I|_{\mathcal{O}_c}\right)(x_1, y_1, x_2, y_2, z) = 0\};$$
$$\mathcal{K}_1 := \{(x_1, y_1, x_2, y_2, z) \in \mathcal{O}_c \mid \mathbf{d}\left(H|_{\mathcal{O}_c}\right)(x_1, y_1, x_2, y_2, z) = 0, \ \mathbf{d}\left(I|_{\mathcal{O}_c}\right)(x_1, y_1, x_2, y_2, z) \neq 0\}.$$

Proposition 2.1. *On a regular symplectic leaf $\mathcal{O}_c$ we have the following characterization for the equilibria:*

$$\mathcal{K}_0 = \mathcal{O}_c \cap (\mathcal{E}_1 \cup \mathcal{E}_3); \quad \mathcal{K}_1 = \mathcal{O}_c \cap \mathcal{E}_2.$$

Proof. Because (2.1) is a Hamiltonian system on a symplectic manifold the condition $\mathbf{d}\left(H|_{\mathcal{O}_c}\right)(e) = 0$ is verified for any equilibrium point $e \in \mathcal{O}_c$.
Let $e_2 \in \mathcal{O}_c \cap \mathcal{E}_2$. Then

$$T_{e_2}\mathcal{O}_c = \{\bar{v} = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5 | \ < \bar{v}, \nabla C(e_2) >= 0\} = \{\bar{v} \in \mathbb{R}^5 | \ v_1 M + v_3 N + v_5 = 0\}.$$

We also have $\mathbf{d}I(e_2) = N\mathbf{d}y_1 - M\mathbf{d}y_2$. Taking, for example, $\bar{v} = (-N, N, M, -M, 0) \in T_{e_2}\mathcal{O}_c$ we have $\mathbf{d}\left(I|_{\mathcal{O}_c}\right)(e_2)(\bar{v}) = M^2 + N^2 \neq 0$, which proves that $\mathbf{d}\left(I|_{\mathcal{O}_c}\right)(e_2) \neq 0$.

For the equilibria e in $\mathcal{E}_1 \cup \mathcal{E}_3$ the condition $\mathbf{d}\left(I|_{\mathcal{O}_c}\right)(e) = 0$ is trivially verified. □

The commutativity of the constants of motion $H|_{\mathcal{O}_c}$ and $I|_{\mathcal{O}_c}$ with respect to the symplectic form $\omega_{\mathcal{O}_c}$ implies that at an equilibrium point $e \in \mathcal{O}_c$ we have

$$\left[\mathbf{D}X_{H|_{\mathcal{O}_c}}(e), \mathbf{D}X_{I|_{\mathcal{O}_c}}(e)\right] = 0,$$

where $\mathbf{D}X_{H|_{\mathcal{O}_c}}(e)$ and $\mathbf{D}X_{I|_{\mathcal{O}_c}}(e)$ are the derivatives of the vector fields $X_{H|_{\mathcal{O}_c}}$ and $X_{I|_{\mathcal{O}_c}}$ at the equilibrium e and consequently $\mathbf{D}X_{H|_{\mathcal{O}_c}}(e)$, $\mathbf{D}X_{I|_{\mathcal{O}_c}}(e)$ are infinitesimally symplectic relative to the symplectic form $\omega_{\mathcal{O}_c}(e)$ on the vector space $T_e\mathcal{O}_c$.

Definition 2.1. *An equilibrium point $e \in \mathcal{K}_0$ is called non-degenerate if $\mathbf{D}X_{H|_{\mathcal{O}_c}}(e)$ and $\mathbf{D}X_{I|_{\mathcal{O}_c}}(e)$ generate a Cartan subalgebra of the Lie algebra of infinitesimal linear transformations of the symplectic vector space $(T_e\mathcal{O}_c, \omega_{\mathcal{O}_c}(e))$. A Cartan subalgebra of the Lie algebra $sp(4,\mathbb{R})$ is a two dimensional commutative sub-algebra which contains an element whose eigenvalues are all distinct.*

It follows that for a non-degenerate equilibrium belonging to $\mathcal{K}_0$ the matrices $\mathbf{D}X_{H|_{\mathcal{O}_c}}(e)$ and $\mathbf{D}X_{I|_{\mathcal{O}_c}}(e)$ can be simultaneously conjugated to one of the following four Cartan sub-algebras

$$\begin{array}{ll} \text{Type 1: } \begin{bmatrix} 0 & 0 & -A & 0 \\ 0 & 0 & 0 & -B \\ A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix} & \text{Type 2: } \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & 0 & 0 & -B \\ 0 & 0 & A & 0 \\ 0 & B & 0 & 0 \end{bmatrix} \\ \text{Type 3: } \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & B & 0 & B \end{bmatrix} & \text{Type 4: } \begin{bmatrix} -A & -B & 0 & 0 \\ B & -A & 0 & 0 \\ 0 & 0 & A & -B \\ 0 & 0 & B & A \end{bmatrix} \end{array} \tag{2.2}$$

where $A, B \in \mathbb{R}$ (see, e.g., [4], Theorems 1.3 and 1.4).
Equilibria of type 1 are called *center-center* with the corresponding eigenvalues for the linearized system: $iA, -\imath A, iB, -\imath B$.
Equilibria of type 2 are called *center-saddle* with the corresponding eigenvalues for the linearized system: $A, -A, \imath B, -\imath B$.
Equilibria of type 3 are called *saddle-saddle* with the corresponding eigenvalues for the linearized system: $A, -A, B, -B$.
Equilibria of type 4 are called *focus-focus* with the corresponding eigenvalues for the linearized system: $A+\imath B, A-\imath B, -A+\imath B, -A-\imath B$.

Theorem 2.2. *We have the following stability behavior for the equilibria in $\mathcal{O}_c \cap \mathcal{E}_1$:*

(i) The equilibrium point $\mathcal{O}_c \cap \mathcal{E}_1 = \{(0,0,0,0,c)\}$ for $c > 0$ is a non-degenerate equilibrium of type focus-focus and consequently unstable.

(ii) The equilibrium point $\mathcal{O}_c \cap \mathcal{E}_1 = \{(0,0,0,0,c)\}$ for $c < 0$ is a non-degenerate equilibrium of type center-center and consequently stable.

Proof. (i) For the linearized systems at the equilibrium $(0,0,0,0,c)$ we have:

$$\mathbf{D}X_{H|_{\mathcal{O}_c}}(0,0,0,0,c) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 \end{bmatrix}$$

and its characteristic polynomial has the non-distinct eigenvalues $\sqrt{c}, \sqrt{c}, -\sqrt{c}, -\sqrt{c}$ and respectively

$$\mathbf{D}X_{I|_{\mathcal{O}_c}}(0,0,0,0,c) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and its characteristic polynomial has the non-distinct eigenvalues $\imath, \imath, -\imath, -\imath$.
To decide the type of stability we need to determine the non-degeneracy of the equilibrium $(0,0,0,0,c)$, i.e. we have to find a linear combination $\mathbf{D}X_{H|_{\mathcal{O}_c}}(0,0,0,0,c) + \alpha\mathbf{D}X_{I|_{\mathcal{O}_c}}(0,0,0,0,c)$, where α is a non-zero real number, that has distinct eigenvalues. The characteristic polynomial of such a linear combination is given by

$$t^4 + (2\alpha^2 - 2c)t^2 + (\alpha^2 + c)^2.$$

After the substitution $t^2 = s$ we obtain the quadratic polynomial

$$s^2 + (2\alpha^2 - 2c)s + (\alpha^2 + c)^2,$$

which has the discriminant $\Delta = -16c\alpha^2 < 0$ and therefore has two distinct complex roots. It follows that the characteristic polynomial $t^4 + (2\alpha^2 - 2c)t^2 + (\alpha^2 + c)^2$ has four distinct complex eigenvalues of the form $A + \imath B, A - \imath B, -A + \imath B, -A - \imath B$ with $A, B \in \mathbb{R}^*$. Consequently, the equilibrium $(0, 0, 0, 0, c)$ for $c > 0$ is a non-degenerate equilibrium of focus-focus type for the dynamics (2.1) and therefore unstable for this dynamics.

Similar computations lead to the proof of (ii). □

Although the equilibrium $(0, 0, 0, 0, 0) \in \mathcal{O}_0$ belongs to $\mathcal{K}_0$, it is a degenerate equilibrium in the sense of Definition 2.1. Indeed, any linear combination $\alpha \mathbf{D} X_{H|_{\mathcal{O}_0}}(0, 0, 0, 0, 0) + \beta \mathbf{D} X_{I|_{\mathcal{O}_0}}(0, 0, 0, 0, 0)$ has the characteristic polynomial $(t^2 + \beta^2)^2$, which has non-distinct eigenvalues. Its stability property can be established using an algebraic method (see [1], [5], [6], [7]). More precisely, the system of algebraic equations

$$H(x_1, y_1, x_2, y_2, z) = H(0, 0, 0, 0, 0), I(x_1, y_1, x_2, y_2, z) = I(0, 0, 0, 0, 0), C(x_1, y_1, x_2, y_2, z) = C(0, 0, 0, 0, 0)$$

has as unique solution the equilibrium $(0, 0, 0, 0, 0)$, leading to the following stability result.

Theorem 2.3. *The equilibrium* $(0, 0, 0, 0, 0)$ *is degenerate and stable with respect to the dynamics* (1.1).

3 Homoclinic orbits

In this section we will give an explicit form of the homoclinic orbits for the unstable equilibria of focus-focus type. This type of equilibria belong to symplectic orbits $\mathcal{O}_c$ with $c > 0$.

In order to compute the homoclinic orbits, we introduce a local system of coordinates around the equilibrium point $e_c = (0, 0, 0, 0, c) \in \mathcal{O}_c$. The local system of coordinates is given by

$$\Phi : \mathbb{R}^5 \to \mathbb{R}^5, \quad (r_1, \theta, y_1, y_2, c) \mapsto \begin{cases} x_1 = r_1 \cos\theta \\ x_2 = r_1 \sin\theta \\ y_1 = y_1 \\ y_2 = y_2 \\ z = c - \frac{1}{2} r_1^2. \end{cases}$$

Freezing the parameter c we obtain the local system of coordinates on the symplectic orbit $\mathcal{O}_c$ around the equilibrium point e_c:

$$\Phi_c : \mathbb{R}^4 \to \mathcal{O}_c \setminus \{e_c\}, \quad (r_1, \theta, y_1, y_2) \mapsto \begin{cases} x_1 = r_1 \cos\theta \\ x_2 = r_1 \sin\theta \\ y_1 = y_1 \\ y_2 = y_2. \end{cases}$$

As we have excluded the equilibrium point e_c we can work under the assumption that $r_1 \neq 0$. The advantage of using polar coordinates in the study of bi-focal homoclinic orbits in four dimensions can be ascertained in [8]. By a straightforward computation we obtain that the reduced system on the symplectic leaf $\mathcal{O}_c$ is given by

$$\begin{cases} \dot{r}_1 = y_1 \cos\theta + y_2 \sin\theta \\ \dot{\theta} = \dfrac{y_2 \cos\theta - y_1 \sin\theta}{r_1} \\ \dot{y}_1 = r_1 \cos\theta \left(c - \frac{1}{2} r_1^2\right) \\ \dot{y}_2 = r_1 \sin\theta \left(c - \frac{1}{2} r_1^2\right). \end{cases} \tag{3.1}$$

Using a continuity argument and the fact that $I(x_1, y_1, x_2, y_2, z) = x_2 y_1 - x_1 y_2$ is a constant of motion we obtain that if there exists a homoclinic it should belong to the connected component of level set

$I^{-1}(I(e_c)) = I^{-1}(0)$ that contains e_c. If a curve $c(t) = (r_1(t), \theta(t), y_1(t), y_2(t))$ is a homoclinic, then it has to be a solution for the system (3.1) and to satisfy the following equation for all t:

$$r_1(t)(y_1(t)\sin\theta(t) - y_2(t)\cos\theta(t)) = 0.$$

This implies that $\dot{\theta}(t) = 0$ and thus $\theta(t) = \theta_0$ constant for all t. By differentiation and substitution we obtain the following second order equation

$$\ddot{r}_1 = r_1\left(c - \frac{1}{2}r_1^2\right).$$

Making the change of variable $r_1 = 2\sqrt{c}\ \tilde{r}_1$ and the time re-parametrization $\sqrt{c}\ t = \tilde{t}$ we obtain the equation

$$\ddot{\tilde{r}}_1(\tilde{t}) = \tilde{r}_1(\tilde{t}) - 2\tilde{r}_1^3(\tilde{t}).$$

It is well known that this second order differential equation has as solutions $\pm\mathrm{cn}(\tilde{t}, 1) = \pm\mathrm{sech}(\tilde{t})$. Consequently, we obtain $r_1(t) = \pm 2\sqrt{c}\ \mathrm{sech}(\sqrt{c}t)$.

Substituting $r_1(t)$ in the expression of the local parametrization Φ_c, and for z in the expression of local parametrization Φ and integrating for y_1 and y_2 in (3.1) we obtain the homoclinic solutions

$$\begin{cases} x_1(t) = \pm 2\sqrt{c}\ \mathrm{sech}(\sqrt{c}t)\cos\theta_0 \\ x_2(t) = \pm 2\sqrt{c}\ \mathrm{sech}(\sqrt{c}t)\sin\theta_0 \\ y_1(t) = \mp 2c\ \mathrm{sech}(\sqrt{c}t)\mathrm{tanh}(\sqrt{c}t)\cos\theta_0 \\ y_2(t) = \mp 2c\ \mathrm{sech}(\sqrt{c}t)\mathrm{tanh}(\sqrt{c}t)\sin\theta_0 \\ z(t) = c(1 - 2\mathrm{sech}^2(\sqrt{c}t)). \end{cases}$$

The above homoclinic orbits, using different parametrization and arguments, have been discussed in [9], [10].

4 Invariant sets and periodic orbits

We will look for invariant sets of the system (1.1) using the technique presented in [3]. We have the following vectorial conserved quantity $\mathbf{F} : \mathbb{R}^5 \to \mathbb{R}^3, \mathbf{F}(p) = (H(p), I(p), C(p))$. In [3], Theorem 2.3, it has been proved that the set $M_{(2)}^{\mathbf{F}} = \{p \in \mathbb{R}^5 |\ \mathrm{rank}\ \nabla\mathbf{F}(p) = 2\}$ is invariant under the dynamics of the system. By direct computation we obtain that $M_{(2)}^{\mathbf{F}} = M_1 \cup M_2$, where

$$M_1 := \left\{\left(x_1, y_1, x_2, -\frac{x_1 y_1}{x_2}, -\frac{y_1^2}{x_2^2}\right) \mid x_2 \neq 0\right\};$$
$$M_2 := \left\{\left(x_1, 0, 0, y_2, -\frac{y_2^2}{x_1^2}\right) \mid x_1 \neq 0\right\}.$$

The union $M_1 \cup M_2$, which is a connected set in $\mathbb{R}^5$, is invariant under the dynamics (1.1), but neither the set M_1 nor the set M_2 are invariant under this dynamics. The vector field corresponding to (1.1) is tangent to the sub-manifold M_1 and the restricted dynamics on M_1 is given by

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = -\frac{x_1 y_1^2}{x_2^2} \\ \dot{x}_2 = -\frac{x_1 y_1}{x_2}. \end{cases} \tag{4.1}$$

We notice that the above dynamical system has two conserved quantities, $f_1, f_2 : M_1 \to \mathbb{R}$, $f_1(x_1, y_1, x_2) = x_1^2 + x_2^2$ and $f_2(x_1, y_1, x_2) = \frac{y_1}{x_2}$. Using these conserved quantities and choosing an initial condition x_1^0, y_1^0, x_2^0 with $x_2^0 \neq 0$ and $y_1^0 \neq 0$ we can explicitly solve the system (4.1):

$$\begin{cases} x_1(t) = x_2^0 \sin\left(\frac{y_1^0}{x_2^0}t\right) + x_1^0\cos\left(\frac{y_1^0}{x_2^0}t\right) \\ y_1(t) = -\frac{y_1^0}{x_2^0}\left(x_1^0\sin\left(\frac{y_1^0}{x_2^0}t\right) - x_2^0\cos\left(\frac{y_1^0}{x_2^0}t\right)\right) \\ x_2(t) = -x_1^0\sin\left(\frac{y_1^0 t}{x_2^0}t\right) + x_2^0\cos\left(\frac{y_1^0}{x_2^0}t\right). \end{cases}$$

Notice that if $y_1^0 = 0$ we obtain as constant solutions the equilibrium points from $\mathcal{E}_2 \subset M_1$. The above solution is defined on time intervals (t_k, t_{k+1}), where $t_k = \frac{x_2^0}{y_1^0}\vartheta + k\pi\frac{x_2^0}{y_1^0}$ with $k \in \mathbb{Z}$ and $\vartheta \in [0, 2\pi)$ is the unique real number such that $\sin\vartheta = \frac{x_2^0}{\sqrt{(x_1^0)^2+(x_2^0)^2}}$ and $\cos\vartheta = \frac{x_1^0}{\sqrt{(x_1^0)^2+(x_2^0)^2}}$. For time values t_k the above solution exits the set M_1 and punctures the set M_2, thus making the union $M_1 \cup M_2$ an invariant set. Although the solution starting from M_1 is not complete, we can construct a complete periodic solution for the initial system (1.1) given by

$$\begin{cases} x_1(t) = x_2^0 \sin\left(\frac{y_1^0}{x_2^0}t\right) + x_1^0 \cos\left(\frac{y_1^0}{x_2^0}t\right) \\ y_1(t) = -\frac{y_1^0}{x_2^0}\left(x_1^0 \sin\left(\frac{y_1^0}{x_2^0}t\right) - x_2^0 \cos\left(\frac{y_1^0}{x_2^0}t\right)\right) \\ x_2(t) = -x_1^0 \sin\left(\frac{y_1^0}{x_2^0}t\right) + x_2^0 \cos\left(\frac{y_1^0}{x_2^0}t\right) \\ y_2(t) = -\frac{y_1^0}{x_2^0}\left(x_2^0 \sin\left(\frac{y_1^0}{x_2^0}t\right) + x_1^0 \cos\left(\frac{y_1^0}{x_2^0}t\right)\right) \\ z(t) = -\frac{(y_1^0)^2}{(x_2^0)^2}. \end{cases}$$

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